

# **Choice and Regularity: Common Consequences in Logic**

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It is well-known that Choice and Regularity are *independent* of each other but have important *common consequences* of logical character (reflection principles, representations of classes by sets, etc.). In my talk, I shall try:

(A) *To explain this phenomenon,*

(B) *To consider relationships between these consequences (and near principles) in detail,*

and besides,

(C) *To consider some arguments related to truth of various principles in set theory.*

All theorems can be proved in ZF, the Zermelo-Fränkel set theory, *minus* Regularity (quite often in some its fragments).

# **Choice and Regularity**

Basic definitions:

A function  $F : X \rightarrow \bigcup X$  is a *choice function* iff  $F(x) \in x$  for all nonempty  $x \in X$ .

A relation  $R \subseteq X \times X$  is *well-founded* iff each nonempty subset  $S \subseteq X$  has an  $R$ -minimal element (i.e.,  $x \in S$  such that  $\neg(y R x)$  for all  $y \in S - \{x\}$ ).

An ordering  $\leq$  is a *well-ordering* of  $X$  iff each nonempty subset  $S \subseteq X$  has a  $\leq$ -least element (i.e.,  $x \in S$  such that  $x \leq y$  for all  $y \in S$ ).

Equivalently,  $\leq$  is linear and well-founded.

**The Axiom of Choice, AC.** *For any set there is a choice function on it.*

There are a number of equivalent principles, the most famous are perhaps Zorn's Lemma (Kuratowski) and

**The Well-Ordering Principle, WO.** *For any set there is a well-ordering on it.*

**Theorem** (Zermelo). *AC is equivalent to WO.*

AC has a deep impact on the universe of set theory by giving as nice consequences, e.g.,

*All cardinals form a well-ordered hierarchy*  
as well as ugly ones, e.g.,

The Banach-Tarski Paradox.

**The Axiom of Regularity, AR.** *Any nonempty set has an  $\in$ -minimal element.*

A set is *well-founded* iff  $\in$  is well-founded on its transitive closure. Then AR is equivalent to the sentence:

*All sets are well-founded*

(and another name of AR is the *Axiom of Foundation*).

To formulate an equivalent principle, recall the *cumulative hierarchy* of sets:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= P(V_\alpha), \\ V_\alpha &= \bigcup_{\beta < \alpha} V_\beta \text{ if } \alpha \text{ is limit.} \end{aligned}$$

**Theorem** (von Neumann). *AR is equivalent to*

$$V = \bigcup_{\alpha \in Ord} V_\alpha.$$

If  $R \subseteq X \times X$  is well-founded, we have:

- (i)  $R$ -Induction
- (ii)  $R$ -Recursion
- (iii) The *rank* function

$$\text{rk}_R : (X, R) \rightarrow (Ord, <)$$

i.e., a strong homomorphism stratifying  $X$  into levels  $X_\alpha$ :

$$X = \bigcup_{\alpha} X_\alpha$$

- (iv) The *transitive collapse*

$$\pi_R : (X, R) \rightarrow (\bigcup_{\alpha} V_\alpha, \in)$$

which is a strong homomorphism allowing to get

**Theorem** (The Mostowski Collapsing Lemma).  
*Any extensional well-founded relation is isomorphic to a unique transitive one (and a unique possible isomorphism is its transitive collapse).*

So, AR has mainly a “simplifying” character: we can use all these nice properties.

Quite often principles have local and global forms. Typically, a *local/global* principle says about sets/classes or one formula/all formulas. Global versions of the previous principles:

**The Global Choice, GC.** *There is a choice function on the universe.*

**The Global Well-Ordering, GWO.** *There is a well-ordering of the universe.*

**The Global Regularity, GR.** *Any nonempty class has an  $\in$ -minimal element.*

(GR is a schema. To formulate GC, we add a new functional symbol. For GWO, we add a new predicate symbol.)



**Lemma.**

1. *GR is equivalent to AR.*
2. *GWO implies GC.*
3. *GC implies AC.*
4.  *$AC + \neg GC$  is consistent.*
5.  *$GC + AR$  implies GWO.*
6.  *$GWO + \neg AR$  is consistent.*
7.  *$\neg AC + AR$  is consistent.*
8.  *$\neg AC + \neg AR$  is consistent.*

(The only hard clause is (4). Later I shall show that one can sharp (5) by replacing “implies” with “is equivalent to” and AR with a weaker principle BF.)

To complete this account, note that Choice *plus* Regularity together can be formulated in a single way:

**The Choice of Minimals, ACM.** *For any set  $X$  there is a choice function  $F$  on  $X$  such that  $F(x) \cap x = \emptyset$  for all nonempty  $x \in X$ .*

**The Global Choice of Minimals, GCM.**  
*There is a choice function  $C$  on  $V$  such that  $C(x) \cap x = \emptyset$  for all nonempty sets  $x$ .*

Clearly,

**Lemma.**

1. ACM is equivalent to AC + AR.
2. GCM is equivalent to GC + AR.

# **Best-Foundedness**

To explicate why Choice (mainly in the strongest form GWO) and Regularity have common consequences, I isolate their “intersection”: a principle (called here Best-Foundedness) which is consistent with negations of both axioms but implies all these consequences.

Let me say that a well-founded relation  $E$  is *best-founded* iff  $\{x : \text{rk}_E(x) = \alpha\}$  is a set for every ordinal  $\alpha$ .

By Replacement, then  $U_\alpha = \{x : \text{rk}_E(x) < \alpha\}$  is also a set for every  $\alpha$ .

*Examples.* The empty relation on a proper class is well- but not best-founded.  $\in$  is best-founded on transitive well-founded sets, and so (by the Mostowski theorem) all extensional well-founded relations are best-founded.

**The Best-Foundedness Axiom, BF.** *There is a best-founded relation on  $V$ .*

(The axiom is in the language with a new predicate symbol.)

**Lemma.**

1. *AR implies BF.*
2. *GWO implies BF.*
3.  *$\text{BF} \vdash \neg \text{AC} \vdash \neg \text{AR}$  is consistent.*

The principle has a number of reformulations (in appropriate languages). Define:

$A$  is *club* iff it is  $\subseteq$ -cofinal in  $V$  and for any  $\subseteq$ -directed  $x \subseteq A$  we have  $\bigcup x \in A$ .

$A$  is a *basis* iff  $\{P^\alpha(x) : x \in A \wedge \alpha \in Ord\}$  is  $\in$ -cofinal in  $V$ .

*Example.* AR is equivalent to any of (1) and (2):

1.  $\{V_\alpha : \alpha \in Ord\}$  is club.
2.  $V_1 (= \{\emptyset\})$  is a basis.

Under BF the sets  $U_\alpha$  play much the same part as the sets  $V_\alpha$  under AR. E.g.,  $\{U_\alpha : \alpha \in Ord\}$  is club.

Moreover,

**Lemma.** *BF is equivalent to any of (1)–(4):*

- 1. There is a function  $F : V \rightarrow \text{Ord}$  such that  $F^{-1}(\alpha)$  is a set for all  $\alpha$ .*
- 2. There is a well-ordered  $\in$ -cofinal in  $V$  class.*
- 3. There is a well-ordered club class.*
- 4. There is a well-ordered basis.*
- 5. There is a well-ordered partition of  $V$  into sets.*

Clause (2) gives a visual notion about BF: intuitively, ordinals of a model show its “height”; then a model witnessing BF is “stretched upward” while a model refuting BF is “inflated in width”.

A similarity: GWO well-orders the whole universe while BF well-orders some its “essential” part (a basis or a club). Moreover, this can be made in a natural way:

**Lemma.** *If there is a well-orderable class that is club (or a basis), then there is such a class which is moreover  $\in$ - and  $\subseteq$ -well-ordered.*

There are less obvious reformulations of BF, one of which (concerning the ordinal definability) I shall give a bit later.

Finally, BF is exactly what is missing in GC to be GWO:

**Theorem.**  *$GC + BF$  is equivalent to GWO.*

(Cf. with (5) of Lemma above.)



# Consequences

Showing that BF works, I shall consider following its consequences:

The existence of Skolem and Scott functions,  
The reflection of formulas at sets,  
The expressibility of the ordinal definability,  
The representability of equivalence classes by sets,

and relationships between them.

Let  $\varphi(u, \dots, x)$  be a formula with the parameters  $u, \dots, x$ .

A function  $A_\varphi$  is a *Skolem function* for  $\varphi$  iff

$$(\exists x) \varphi(u, \dots, x) \rightarrow \varphi(u, \dots, A_\varphi(u, \dots)).$$

Similarly, let me say that a function  $B_\varphi$  is a *Scott function* for  $\varphi$  iff

$$(\exists x) \varphi(u, \dots, x) \rightarrow (\exists x \in B_\varphi(u, \dots)) \varphi(u, \dots, x)$$

and

$$(\forall x \in B_\varphi(u, \dots)) \varphi(u, \dots, x).$$

Thus  $A_\varphi$  chooses a single point from the class  $\{x : \varphi(u, \dots, x)\}$ :

$$A_\varphi(u, \dots) \in \{x : \varphi(u, \dots, x)\}$$

while  $B_\varphi$  separates from this class its subset

$$B_\varphi(u, \dots) \subseteq \{x : \varphi(u, \dots, x)\}$$

such that the set  $B_\varphi(u, \dots)$  is nonempty whenever the class  $\{x : \varphi(u, \dots, x)\}$  so is.

*Remark.* Scott was probably first who noted that such functions can be used instead of Skolem functions in absence of AC.

Consider the following schemas (in extended languages):

**The Skolem Principle, Sk.** *For any formula there is a Skolem function.*

**The Scott Principle, Sc.** *For any formula there is a Scott function.*

$Sk_\varphi$  and  $Sc_\varphi$  denote the instances of these schemas.

**Lemma.**

1.  $Sk_\varphi$  implies  $Sc_\varphi$ .
2.  $AC + Sc_\varphi$  implies  $Sk_\varphi$ .
3. BF implies Sc.
4.  $AC + BF$  implies Sk.

Via coding, one can formulate global variants of Skolem and Scott functions (uniformly for all formulas). A global Skolem function acts like a choice function on definable classes while a global Scott function separates subsets from them. Let  $GSk$  and  $GSc$  denote the global variants of  $Sk$  and  $Sc$ .

**Lemma.**

1.  $GSk$  is equivalent to  $GC + GSc$ .
2.  $GSk + BF$  is equivalent to  $GWO$ .

I need Scott (or Skolem) functions mainly to have Reflection.

A class  $M$  *reflects* a formula  $\varphi(x, \dots)$  iff for all  $x, \dots \in M$

$$\varphi^M(x, \dots) \leftrightarrow \varphi(x, \dots).$$

**The Reflection Principle, RP.** *Each formula is reflected at some set.*

(RP is a schema,  $\text{RP}_\varphi$  are instances.)

It follows from RP that each true formula has a set model.

Of course, the principle holds for finitely many formulas as well. Let me rewrite it as follows: If  $\Gamma$  is a finite set of formulas, then there is a set  $M$  such that

$$M \prec_{\Gamma} V.$$

Thus RP is a local variant of the Löwenheim-Skolem Theorem. But unlike it, RP can be proved *inside* (some) set theory:

**Theorem.** *Sc implies RP.*

(Take a *Scott hull*.)

Moreover, Sc gives a club class of reflecting sets, and BF gives a club class of reflecting sets of form  $U_{\alpha}$ .

*Remarks.* 1. Without Choice, we know nothing about the size of submodels.

2. The full Löwenheim-Skolem Theorem (without an evaluation of the cardinality) can be obtained in the same way as a metatheorem.



We consider two applications of Reflection. The first concerns the finite axiomatizability:

Let us call a theory *sufficiently rich* iff it admits a coding. (E.g., ZF minus Infinity so is).

**Proposition.** *Let  $T$  be sufficiently rich consistent theory and  $T \vdash RP$ . Then  $T$  is not finitely axiomatizable.*

(Apply the Second Incompleteness Theorem.)

*Examples.* The theory consisting of Union, Power Set, Replacement, and Best-Foundedness is not finitely axiomatizable. The same for any its consistent extension (e.g., ZF). On the other hand, in the Zermelo set theory  $Z$  (which is finitely axiomatizable),  $RP$  is not provable.

Another application of RP: the description of ordinal-definable sets inside of set theory.

A class is *ordinal-definable* iff it is of the form  $\{u : \varphi(u, \alpha, \dots)\}$  for some formula  $\varphi$  where all  $\alpha, \dots$  are ordinals.  $OD$  is the class of all ordinal-definable sets.  $\text{cl}$  is the closure under Gödel operations.

A well-known fact: AR implies

$$OD = \text{cl}(\{V_\alpha : \alpha \in \text{Ord}\}).$$

It follows that  $OD$  is well-orderable and club (and moreover, the largest inner model of ZF with a global well-ordering definable via  $\in$ ).

We sharp:

**Theorem.** *BF implies*

$$OD = \text{cl}(\{U_\alpha : \alpha \in \text{Ord}\}).$$

(Use RP to prove  $\subseteq$ .)

**Corollary.** *BF holds iff OD is well-orderable and club.*

Thus again (like the characteristic of GWO via BF and GC) we sharp an old result of form

$$\Gamma + \text{AR} \text{ implies } \Delta$$

by a new result of form

$$\Gamma + \text{BF} \text{ is equivalent to } \Delta$$

(where  $\Gamma$  and  $\Delta$  are some sets of formulas).  
This supports a naturality of BF.

As the last interesting consequence of BF, consider representations of equivalence classes by sets.

Let  $\varphi(x, y)$  define an equivalence:

$$\varphi(x, y) \wedge \varphi(y, z) \rightarrow \varphi(y, x) \wedge \varphi(x, z).$$

A function  $F_\varphi$  *represents* the equivalence defined by  $\varphi$  iff

$$\varphi(x, y) \leftrightarrow F_\varphi(x) = F_\varphi(y).$$

**The Representation of Classes Principle, RC.** *For any equivalence formula there is a representing function.*

(RC is a schema,  $RC_\varphi$  are instances.)

Of course,

$\text{Sc}_\varphi$  implies  $\text{RC}_\varphi$

since any Scott function for  $\varphi$  represents the equivalence in a “natural way”. But unlike Scott functions,  $F_\varphi(x)$  does not meet necessarily the equivalence class  $\{y : \varphi(x, y)\}$ .

Sometimes AC suffices for some instances of RC:

*Example.* If  $\varphi$  expresses the same cardinality, then AC implies  $\text{Sk}_\varphi$  and so  $\text{RC}_\varphi$ .

Moreover,

**Theorem.** *Let  $\varphi$  define an equivalence. Then  $\text{AC} + \text{RP}_\varphi$  implies  $\text{Sk}_\varphi$ .*

**Corollary.**  *$\text{AC} + \text{RP}$  implies RC.*

**Question.** *Is any of the following implications provable:*

1. *AC implies Sc?*
2. *AC + Sc implies Sk?*
3. *Sc implies BF?*
4. *GC implies BF?*
5. *GC implies RC? if No for (1)–(4).*

**Conjecture.** *No for all (1)–(5).*

(Partial results.)

To prove consistency results of such kind, I develop a method of construction of models via *automorphism filters* (a generalization of well-known permutation model method). Two main technical obstacles arising without Regularity (or with a proper class of atoms):

- (i) Replacement,
- (ii) The Transfer Theorem (Jech–Sochor).

# **Truth and Interpretability Strength**

ZFC is highly incomplete: there are very natural set-theoretical questions independent of it, the most famous of which is perhaps the Continuum Hypothesis (which is really a *third order arithmetical* sentence).

To complete ZFC, it was proposed a number of principles of various kind having important consequences: e.g.,

Large Cardinal Axioms,  
The Axiom of Determinacy (Mycielski),  
Proper Forcing Axioms (Shelah),  
The  $\Omega$  Conjecture (Woodin),  
Generic Large Cardinals (Foreman),  
The Inner Model Hypothesis (Sy Friedman),  
etc.

Is there a general criterion for rejecting/accepting such a principle as a *true* axiom about *all* sets?



Let me point out a simple criterion indicating some principles as *surely wrong*. An idea: since all mathematical objects are sets, an ideal TST (“True Set Theory”) must capture all possible theories. Hence having some  $T \subseteq \text{TST}$  and examining a new principle  $\Gamma$ , we must reject  $\Gamma$  if it restricts this possibility:

*If  $T + \Gamma$  loses the interpretability strength of  $T$  then  $\Gamma$  is wrong.*

Here: An extension  $T_1 \supseteq T$  of a theory  $T$  *loses the interpretability strength* of  $T$  iff there is  $T_2 \supseteq T$  non-interpretable in all  $T_3 \supseteq T_1$ . Otherwise  $T_1$  extends  $T$  *without loss of the interpretability strength*.

*Example.* “All sets are constructible”

$$V = L$$

is a nice axiom since it looks “empirically complete”, but it loses the interpretability strength of ZFC: Under  $V = L$ , there is no measurable cardinals, even in inner models. So, it is wrong. May be so is “All sets are ordinal-definable”

$$V = OD?$$

or even “All sets are well-founded”

$$V = \bigcup_{\alpha} V_{\alpha}?$$

(See Question below.)

To find a criterion indicating some principles as *surely true* is much more hard. (Of course, if  $\varphi$  is surely wrong then  $\neg\varphi$  is surely true but too *noneffective* as a rule. E.g., cf. “there exists a nonconstructible set” with “ $0^\sharp$  exists”.) Advancing in the same way, we can describe only *possibly true* principles:

*If  $T + \Gamma$  extends  $T$  without loss of the interpretability strength and does not interpret in  $T$ , then  $\Gamma$  can be true.*

Such candidates for being true cannot be really true all together because contradict to each other. But some of them are concordant:

*Example.* Current large cardinal axioms form a well-ordered hierarchy. (An empirical fact; Woodin offers a partial explication.)

*Remark.* Criteria based on interpretability are perhaps most important but not sufficient. E.g., put  $T$  be

$$ZF - \text{Infinity} + \neg \text{Infinity} + \text{Con}(\text{ZFC}).$$

ZFC is interpretable in  $T$ , but I think this theory (in which infinite objects do not exist) is not a correct theory of *all* sets. Likewise for extensions of ZFC by large cardinals. (An explication is beyond my talk.)

Among the axioms of ZFC, only Extensionality (AE) and Regularity have an “impoverishing” character (since forbid sets of certain structure); a character of AC is unclear; and other axioms have an “enriching” character (since permit to construct new sets). Does this “impoverishment” really decrease the interpretability strength? Under GWO, we have easily the answer No:

**Lemma.** *ZF – AE – AR + GWO can be extended by AE + AR without loss of the interpretability strength.*

**Question.** *Can one extend without loss of the interpretability strength:*

1. *ZF – AE – AR by BF?*
2. *ZF – AE – AR + BF by AE + AR?*
3. *ZF by GWO?*